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Existence of Periodic Solutions for Recurrent Cellular Neural Networks with Distributed Delays*

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Abstract: By means of the Mawhin continuation theorem, the existence of periodic solutions for recurrent neural networks is studied. We assume that the behaved function is located in a strip region, and the activation function is in a region between two linear functions.

Keywords: recurrent cellular neural networks; periodic solution; coincidence degree

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1 Introduction

Cellular neural networks (CNNs) are introduced by Chua and Yang. They have many applications in optimization, associative memories, and signal processing. Recently, they have been studied in the literature[1-3]. Because neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, there will be a distribution of propagation delays[4]. Recently, recurrent neural networks have been widely studied. The Hopfield neural networks and cellular neural networks[5] are representative of them.

In the literature[6], the authors proposed a class of recurrent neural networks as follows

$$x_i'(t)=-f_i(x_i(t))+\sum_{j=1}^{n}a_{ij}(t)g_j(x_j(t))+\sum_{j=1}^{n}b_{ij}(t)g_j(x_j(t-\tau_{ij}(t)))+I_i(t),\quad t>0,$$

$$x_i(t)=\phi_i(t)\neq 0,\quad t\in[-\tau,0],\quad i=1,2,\cdots,n,$$

where n denotes the number of units in neural networks, $x_i(t)$ is the state function of the ith unit, $a_{ij}(t)$ and $b_{ij}(t)$ denote the strength of connectivity between the cell i and j at time t, and they are both continuous ω-periodic functions, $f(x)=(f_1(x_1),f_2(x_2),\cdots,f_n(x_n))^T$ is an appropriately behaved function, $g(x)=(g_1(x_1),\cdots,g_n(x_n))^T$ is a nonlinear vector-valued activation function, $I(t)=(I_1(t),I_2(t),\cdots,I_n(t))^T$ is a continuous ω-periodic vector-valued input function, and $0\leq\tau_{ij}(t)\leq\tau$ is a time delay required in processing and transmitting a signal from the jth cell to the ith cell at time t.

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The aim of this paper is to investigate the existence of periodic solution for the networks

$$\begin{aligned} x_i'(t) &= -f_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t) g_j(x_j(t-\tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{n} c_{ij}(t) \int_0^{\infty} k_{ij}(s) h_j(x_j(t-s)) \mathrm{d}s + I_i(t), \quad t \in [0,+\infty), \quad i = 1,2,\cdots,n. \quad (1) \end{aligned}$$

In (1), f_i and g_i are continuous functions, c_{ij} is a continuous ω-periodic connection weight function, h_j is an activation function of signal transmission, and k_{ij} is a delay kernel function. The initial conditions associated with (1) are $x_i(s) = \phi_i(s) \neq 0$, $s \in (-\infty, 0]$, where $\phi_i \in C((-\infty,0],\mathbf{R})$, $i = 1,2,\cdots,n$.

Throughout this paper, we assume that:

(H$_1$) There exist positive constants α_i', β_i', α_i, β_i and $\beta_i' < \beta_i$ $(i = 1,2,\cdots,n)$ such that

$$f_i(0) = 0, \quad \alpha_i'|u| - \beta_i' \le f_i(u) \le \alpha_i|u| + \beta_i, \quad u \in \mathbf{R}.$$

(H$_2$) There exist positive constants σ_j, θ_j such that

$$|g_j(u)| \le \sigma_j|u| + \theta_j, \quad u \in \mathbf{R}, \quad j = 1,2,\cdots,n.$$

(H$_3$) There exist positive constants $Q_j > 0$ such that

$$|h_j(u)| \le Q_j, \quad u \in \mathbf{R}, \quad j = 1,2,\cdots,n.$$

(H$_4$) $\tau_{ij}(t)$ $(i,j = 1,2,\cdots,n)$ are continuously differentiable ω-periodic functions defined on $[0,+\infty)$ and

$$\inf_{t\in[0,+\infty)} \big(1 - \tau_{ij}'(t)\big) > 0.$$

(H$_5$) The delay kernels $k_{ij}(s) : [0,\infty) \to [0,\infty)$ are continuous, and

$$\int_0^{\infty} k_{ij}(s)\mathrm{d}s = k_{ij} > 0, \quad i,j = 1,2,\cdots,n.$$

For the sake of convenience, we introduce the following notations

$$a_{ij}^+ = \max_{0\le t\le\omega} |a_{ij}(t)|, \quad b_{ij}^+ = \max_{0\le t\le\omega} |b_{ij}(t)|, \quad c_{ij}^+ = \max_{0\le t\le\omega} |c_{ij}(t)|, \quad I_i^+ = \max_{0\le t\le\omega} |I_i(t)|,$$

$$|\widetilde{I}_i| = \frac{1}{\omega}\int_0^{\omega} |I_i(t)|\mathrm{d}t, \quad m_{ij} = \Big(\max_{0\le t\le\omega} \frac{1}{1-\tau_{ij}'(t)}\Big)^{\frac{1}{2}}, \quad i,j = 1,2,\cdots,n.$$

2 Preliminaries

Firstly, we introduce the Mawhin continuation theorem[7] as follows:

Lemma 2.1 Let X and Y be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that

(a) For each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;

(b) For each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;

(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

3 Existence of periodic solutions

Theorem 3.1 Assume (H_1)–(H_5) hold, and

$$\alpha_i'(1-\alpha_i\omega)-\sigma_i\sum_{j=1}^{n}(1+\alpha_j'\omega)(a_{ji}^{+}+b_{ji}^{+}m_{ji})>0,$$

$$\alpha_i'-\sum_{j=1}^{n}(a_{ji}^{+}+b_{ji}^{+})\sigma_i>0,\quad i=1,2,\cdots,n,$$

then the network (1) has at least one ω-periodic solution.

Proof Let

$$X=Z=\{x(t)\in C(\mathbf{R},\mathbf{R}^n):x(t)=x(t+\omega),\ x=(x_1,x_2,\cdots,x_n)^T\}.$$

Then X and Z are both Banach spaces with the norm

$$\|x\|_2=\Big(\int_0^{\omega}|x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}.$$

Let $L:\operatorname{Dom}L\subset X\to Z,\ Lx=x',\ x\in\operatorname{Dom}L\subset X$, where $\operatorname{Dom}L=C^1(\mathbf{R},\mathbf{R}^n)\cap X$ and $N:X\to Z$,

$$\begin{aligned}Nx=&\Big(-f_i\big(x_i(t)\big)+\sum_{j=1}^{n}a_{ij}(t)g_j\big(x_j(t)\big)+\sum_{j=1}^{n}b_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big)\\&+\sum_{j=1}^{n}c_{ij}(t)\int_0^{\infty}k_{ij}(s)h_j\big(x_j(t-s)\big)\mathrm{d}s+I_i(t)\Big)_{n\times 1}.\end{aligned}$$

Obviously, $\operatorname{Im}L$ is closed in Z and L is a Fredholm mapping of index zero. Define two projectors P and Q as

$$Px=\frac{1}{\omega}\int_0^{\omega}x(t)\mathrm{d}t,\quad x\in X,\qquad Qz=\frac{1}{\omega}\int_0^{\omega}z(t)\mathrm{d}t,\quad z\in Z.$$

So, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$.

Corresponding to the operator equation $Lx=\lambda Nx,\ \lambda\in(0,1)$, we have

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}=&\lambda\Big[-f_i\big(x_i(t)\big)+\sum_{j=1}^{n}a_{ij}(t)g_j\big(x_j(t)\big)+\sum_{j=1}^{n}b_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big)\\&+\sum_{j=1}^{n}c_{ij}(t)\int_0^{\infty}k_{ij}(s)h_j\big(x_j(t-s)\big)\mathrm{d}s+I_i(t)\Big],\quad i=1,2,\cdots,n.\end{aligned}\tag{2}$$

Suppose that $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T\in X$ is a solution to (2) for some $\lambda\in(0,1)$.

Multiplying both sides of (2) by x_i' and integrating over $[0,\omega]$, we obtain

$$\begin{aligned}\int_0^\omega |x_i'|^2\mathrm{d}t \le\ & \alpha_i\|x_i'\|_2\|x_i\|_2+\beta_i\sqrt{\omega}\,\|x_i'\|_2+\sum_{j=1}^n a_{ij}^+\sigma_j\|x_i'\|_2\|x_j\|_2\\ &+\sum_{j=1}^n a_{ij}^+\theta_j\sqrt{\omega}\,\|x_i'\|_2+I_i^+\sqrt{\omega}\,\|x_i'\|_2+\sum_{j=1}^n b_{ij}^+\sigma_j m_{ij}\|x_i'\|_2\|x_j\|_2\\ &+\sum_{j=1}^n b_{ij}^+\theta_j\sqrt{\omega}\,\|x_i'\|_2+\sum_{j=1}^n c_{ij}^+k_{ij}Q_j\sqrt{\omega}\,\|x_i'\|_2. \end{aligned} \tag{3}$$

From (3), it follows that

$$\begin{aligned}\|x_i'\|_2 \le\ & \alpha_i\|x_i\|_2+\sum_{j=1}^n\big(a_{ij}^+\sigma_j+b_{ij}^+\sigma_j m_{ij}\big)\|x_j\|_2+\Big(\beta_i\sqrt{\omega}+\sum_{j=1}^n a_{ij}^+\theta_j\sqrt{\omega}\\ &+\sum_{j=1}^n b_{ij}^+\theta_j\sqrt{\omega}+\sum_{j=1}^n c_{ij}^+k_{ij}Q_j\sqrt{\omega}+I_i^+\sqrt{\omega}\Big),\quad i=1,2,\cdots,n.\end{aligned}$$

Integrating both sides of (1) over $[0,\omega]$, we obtain

$$\begin{aligned}\int_0^\omega f_i\big(x_i(t)\big)\mathrm{d}t =\ & \sum_{j=1}^n\int_0^\omega a_{ij}(t)g_j\big(x_j(t)\big)\mathrm{d}t+\sum_{j=1}^n\int_0^\omega b_{ij}(t)g_j\big(x_j(t-\tau_{ij}(t))\big)\mathrm{d}t\\ &+\int_0^\omega I_i(t)\mathrm{d}t+\sum_{j=1}^n\int_0^\omega c_{ij}(t)\int_0^\infty k_{ij}(s)h_j\big(x_j(t-s)\big)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Then, there exists some $\xi\in(0,\omega)$ such that

$$\begin{aligned}f_i\big(x_i(\xi)\big) \le\ & \frac{1}{\omega}\sum_{j=1}^n a_{ij}^+\int_0^\omega\big(\sigma_j|x_j(t)|+\theta_j\big)\mathrm{d}t\\ &+\frac{1}{\omega}\sum_{j=1}^n b_{ij}^+\int_0^\omega\big(\sigma_j|x_j(t-\tau_{ij}(t))|+\theta_j\big)\mathrm{d}t+\sum_{j=1}^n c_{ij}^+k_{ij}Q_j+I_i^+. \end{aligned}\tag{4}$$

Applying (H_1) to inequality (4) and by the Hölder inequality, we have

$$\begin{aligned}\alpha_i'|x_i(\xi)| \le\ & \frac{1}{\sqrt{\omega}}\sum_{j=1}^n\big(a_{ij}^++b_{ij}^+m_{ij}\big)\sigma_j\|x_j\|_2\\ &+\sum_{j=1}^n a_{ij}^+\theta_j+\sum_{j=1}^n b_{ij}^+\theta_j+\sum_{j=1}^n c_{ij}^+k_{ij}Q_j+I_i^++\beta_i'.\end{aligned}$$

For any $t\in[0,\omega]$, it follows that

$$|x_i(t)|\le|x_i(\xi)|+\int_0^\omega|x_i'(t)|\mathrm{d}t\le|x_i(\xi)|+\sqrt{\omega}\,\|x_i'\|_2,\quad i=1,2,\cdots,n.$$

Consequently

$$\alpha_i'\|x_i\|_2 = \alpha_i'\Big(\int_0^\omega |x_i(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \le \alpha_i'\sqrt{\omega}\max_{0\le t\le\omega}|x_i(t)| \le \alpha_i'\sqrt{\omega}\,|x_i(\xi)| + \alpha_i'\omega\|x_i'\|_2.$$

Then

$$\alpha_i'(1-\alpha_i\omega)\|x_i\|_2 \le (1+\alpha_i'\omega)\sum_{j=1}^n (a_{ij}^+ + b_{ij}^+ m_{ij})\sigma_j\|x_j\|_2 + R_i^*, \tag{5}$$

where

$$\begin{aligned} R_i^* \equiv\ & \sqrt{\omega}\Big(\sum_{j=1}^n a_{ij}^+\theta_j + \sum_{j=1}^n b_{ij}^+\theta_j + \sum_{j=1}^n c_{ij}^+ k_{ij} Q_j + I_i^+ + \beta_i'\Big) \\ & +\alpha_i'\omega\Big(\beta_i\sqrt{\omega} + \sum_{j=1}^n a_{ij}^+\theta_j\sqrt{\omega} + \sum_{j=1}^n b_{ij}^+\theta_j\sqrt{\omega} + \sum_{j=1}^n c_{ij}^+ k_{ij} Q_j\sqrt{\omega} + I_i^+\sqrt{\omega}\Big). \end{aligned}$$

From (5), it follows that

$$\sum_{i=1}^n \alpha_i'(1-\alpha_i\omega)\|x_i\|_2 \le \sum_{i=1}^n\Big((1+\alpha_i'\omega)\sum_{j=1}^n (a_{ij}^+ + b_{ij}^+ m_{ij})\sigma_j\|x_j\|_2 + R_i^*\Big).$$

Consequently

$$\sum_{i=1}^n \|x_i\|_2 \le H^*,$$

where

$$H^* \equiv \frac{\sum\limits_{i=1}^n R_i^*}{\min\limits_{1\le i\le n}\Big[\alpha_i'(1-\alpha_i\omega) - \sigma_i\sum\limits_{j=1}^n(1+\alpha_j'\omega)(a_{ji}^+ + b_{ji}^+ m_{ji})\Big]}.$$

Clearly, H^* is independent of λ. Denote $M^* = H^* + F$, where $F > 0$ is sufficiently large so that

$$\min_{1\le i\le n}\Big[\alpha_i' - \sum_{j=1}^n(a_{ji}^+ + b_{ji}^+)\sigma_i\Big] * \frac{M^*}{\sqrt{\omega}} - \sum_{i=1}^n\sum_{j=1}^n\big[(a_{ij}^+ + b_{ij}^+)\theta_j + c_{ij}^+ Q_j k_{ij}\big] - \sum_{i=1}^n\big(\beta_i' + |\widetilde{I}_i|\big) > 0.$$

Now we define

$$\Omega = \Big\{x = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T \in X\,\Big|\,\sum_{i=1}^n \|x_i\|_2 < M^*\Big\}.$$

Then Ω satisfies condition (a) in Lemma 2.1. Let $(x_1, x_2, \cdots, x_n)^T \in \partial\Omega\bigcap \mathrm{Ker}\,L$, $(x_1, x_2, \cdots, x_n)^T$ is a constant vector in $\mathbf{R}^n$ with

$$\big\|(x_1, x_2, \cdots, x_n)\big\|_0 \equiv |x_1| + |x_2| + \cdots + |x_n| = \frac{M^*}{\sqrt{\omega}}.$$

Then

$$
\begin{aligned}
\left\|QN(x_1,x_2,\cdots,x_n)\right\|_0 \geq\ & \sum_{i=1}^{n}|f_i(x_i)| - \sum_{i=1}^{n}\Big|\sum_{j=1}^{n}\left(a_{ij}^{+}g_j(x_j)+b_{ij}^{+}g_j(x_j)\right)\Big| \\
& -\frac{1}{\omega}\sum_{i=1}^{n}\int_0^{\omega}|I_i(t)|\mathrm{d}t - \sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}^{+}h_j(x_j)k_{ij}| \\
\geq\ & \min_{1\leq i\leq n}\Big[\alpha_i' - \sum_{j=1}^{n}(a_{ji}^{+}+b_{ji}^{+})\sigma_i\Big]\sum_{i=1}^{n}|x_i| \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(a_{ij}^{+}+b_{ij}^{+})\theta_j + c_{ij}^{+}Q_jk_{ij}\right] - \sum_{i=1}^{n}\left(\beta_i' + |\widetilde{I}_i|\right) > 0.
\end{aligned}
$$

This implies condition (b) of Lemma 2.1.

Define $\Psi : \operatorname{Ker} L \times [0,1] \to X$ by

$$
\begin{aligned}
\Psi(x_1,x_2,\cdots,x_n,\mu) =\ & -\mu\Big(\frac{1}{\omega}\int_0^{\omega}f_1(x_1)\mathrm{d}t,\ \frac{1}{\omega}\int_0^{\omega}f_2(x_2)\mathrm{d}t,\ \cdots,\ \frac{1}{\omega}\int_0^{\omega}f_n(x_n)\mathrm{d}t\Big)^T \\
& + (1-\mu)QN(x_1,x_2,\cdots,x_n)^T,
\end{aligned}
$$

$$
\begin{aligned}
\left\|\Psi(x_1,x_2,\cdots,x_n,\mu)\right\|_0 \geq\ & \sum_{i=1}^{n}|f_i(x_i)| - \sum_{i=1}^{n}\Big[\sum_{j=1}^{n}(a_{ij}^{+}+b_{ij}^{+})(\sigma_j|x_j|+\theta_j) + \sum_{j=1}^{n}c_{ij}^{+}Q_jk_{ij} + |\widetilde{I}_i|\Big] \\
\geq\ & \min_{1\leq i\leq n}\Big[\alpha_i' - \sum_{j=1}^{n}(a_{ji}^{+}+b_{ji}^{+})\sigma_i\Big]\sum_{i=1}^{n}|x_i| \\
& -\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(a_{ij}^{+}+b_{ij}^{+})\theta_j + c_{ij}^{+}Q_jk_{ij}\right] - \sum_{i=1}^{n}\left(\beta_i' + |\widetilde{I}_i|\right) > 0.
\end{aligned}
$$

Hence, using the homotopy invariance theorem, we obtain

$$
\deg\left(QN(x_1,x_2,\cdots,x_n)^T,\ \Omega\cap\operatorname{Ker} L,\ (0,0,\cdots,0)^T\right) \neq 0.
$$

By Lemma 2.1, (1) has at least one ω-periodic solution.

4 Conclusions

By using the Mawhin continuation theorem, derived in this paper are some new sufficient conditions ensuring the existence of periodic solutions for recurrent cellular neural networks with distributed delays.

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具分布时滞的周期运动细胞神经网络周期解的存在性

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摘 要: 运用Mawhin连续性定理研究具分布时滞的周期运动细胞神经网络周期解的存在性，假设行为函数位于一带型区域内，激活函数位于两线性函数所夹的区域内。

关键词: 周期运动细胞神经网络；周期解；迭合度